of high dielectric constant which has been discussed in Fig. 4.

Finally, for completeness, a somewhat different case is treated in Fig. 6. The substrate again is of the plastic type with low dielectric constant of $\epsilon_s = 2.35$. The substrate thickness is now $h_s = 0.79$ mm. In contrast to the cases investigated before, the width w_1 of strip 1 is kept constant at $w_1 \cong 3h_s$ which is equivalent to a characteristic impedance of strip 1 of approximately 50 Ω. With the width w_1 of strip 1 increased by a factor of 3 compared to Fig. 5, the gap capacitance C_g and the stray capacitance C_{s1} should be larger by about the same factor as is indeed confirmed by inspection of the figures. The change-over of the stray element C_{s1} to inductive values does not occur in the range of widths w_2 examined in Fig. 6(b) due to the reduction of the line inductance of line 1 with increased width, and as a result of the larger line capacitance. Likewise, the pronounced capacitive behavior of the structure is clearly visible from the relatively high values of stray capacitance C_{s2} in Fig. 6(c).

III. Conclusion

The elements of the equivalent pi-circuit of the asymmetrical series gap in microstrip and suspended substrate lines have been computed numerically for a wide range of geometries and two representative substrate materials. The results show a behavior of the asymmetric gap which is different from the case of equal widths of the involved strips and which is seen to tend towards that of the impedance step if end-to-end coupling between the strips is tight. Physical understanding, agreement with the results of other authors for the case of equal widths, as well as the asymptotic behavior and values of the computed equivalent circuit data confirm their validity. The information presented in graphical form was not available up to now and is thought to be useful for MIC design and measurement purposes.

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Evaluation of the Integrals Occurring in the Study of Circular Microstrip Disk

KOHEI HONGO AND MASARU TAKAHASHI

***Abstract*—We have derived rigorous series expressions for the integrals which appear in the study of a circular microstrip disk separated from the grounded dielectric substrate when the problem is formulated by the method of dual integral equation or using Kobayashi potentials. The validity of the approximate expression of the integrals for small separation is verified numerically.**

I. Introduction

Recently, the electrostatic problem of a circular parallel plate condenser filled with dielectric has been of interest again because it has application as components of microstrip and antenna circuits. Bokar and Yang [1] studied this problem using the dual integral equation, and shortly later, Chew and Kong [2], starting from a similar approach, have derived the limiting values of capacitance as the separation approaches zero. The method of dual integral equation is closely related with the method of Kobayashi potential [3] developed by Kobayashi and Nomura [4]. The basic idea of these methods is to reduce the problem to a set of linear equations, and the crucial point of the methods is to calculate the matrix elements which are given by infinite integrals, including Bessel functions, as integrands. An analytical calculation of these kinds of integrals was first carried out by Nomura associated with a circular parallel capacitor located in an empty space [5]. But it is found that his result was incomplete, although his approach is very general. Recently, Chew and Kong [2] met with similar integrals which give matrix elements and tried to derive approximate solutions using much mathematical manipulation. However, their calculation was restricted to the first two terms of the power series expansion of the integrals.

The purpose of this article is to show that the infinite integrals can be calculated analytically through a rather straight-forward manner, which are valid for both large separation and small separation. When the separation is very small, approximate ex-

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plicit expressions up to order q^4 are presented. It is found that the approximate expression truncated after the first two terms for diagonal elements gives a precise result even for a rather large separation, while the range of the validities of the approximate solutions for off-diagonal elements is very restrictive. So it seems to be difficult to derive an explicit asymptotic expression for the capacitance using the present or a related approach.

II. Evaluation of the Integrals

As shown in Bokar and Yang [1] or Chew and Kong [2], the potential for a circular microstrip disk of radius a separated from a ground plane by a dielectric material with relative permittivity ϵ_r is given by

$$\Phi_1 = \sum_{n=0}^{\infty} A_n \int_0^{\infty} J_0\left(\frac{r}{a}\xi\right) G(\xi, q) \sinh q\xi \frac{J_{2n+\kappa}(\xi)}{\xi^{\kappa}} \cdot \exp\left[-\frac{z-d}{a}\xi\right] d\xi. \quad (1)$$

For $z>d$ and for $0<z<d$ it is given by

$$\Phi_2 = \sum_{n=0}^{\infty} A_n \int_0^{\infty} J_0\left(\frac{r}{a}\xi\right) G(\xi, q) \frac{J_{2n+\kappa}(\xi)}{\xi^{\kappa}} \sinh\left(\frac{z}{a}\xi\right) d\xi \quad (2)$$

where $q = d/a$ (d: separation of disk from ground plane) is a normalized separation of a disk which is charged to potential V, and $G(\xi, q) = [\sinh(q\xi) + \epsilon_r \cosh(q\xi)]^{-1}$. In this method, there is a freedom for choice of the value κ as described in the literature [1], [2]. When the separation is small, it is convenient to choose $\kappa = 1$. Expansion coefficients A_n are determined from an infinite system of linear equations

$$\sum_{m=0}^{\infty} \kappa(2n+1, 2m+1; \epsilon_r) A_m = E_n \qquad (n = 0, 1, 2, \cdots) \quad (3)$$

where

$$K(n, m; \epsilon) = \int_0^{\infty} G(\xi, q) \sinh(q\xi) \frac{J_n(\xi) J_m(\xi)}{\xi^2} d\xi \quad (4)$$

$$E_n = \begin{cases} 0, & n \geq 1 \\ \bar{V}/2, & n = 0. \end{cases} \quad (5)$$

The capacitance of the disk is shown to be obtained from the expression $C = \epsilon_0 \pi a A_0 / V$.

As a preliminary step to obtain the expression for $K(n, m; \epsilon_0)$ we consider integral defined by

$$H(\mu, \nu; \chi) = \int_0^{\infty} \frac{J_\mu(\xi) J_\nu(\xi)}{\xi^2} \exp(-\chi\xi) d\xi. \quad (6)$$

The solution for the integral has been derived by Nomura [5], but his result seemed to be incomplete for a part of the contribution from simple poles is missing, so we will here recalculate the above integral. Substituting series expression for the product of two Bessel functions and carrying out the integration term by term, we get

$$H(\mu, \nu; \chi) = \frac{1}{2} \cdot \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\mu+\nu+2l-1) \Gamma(\mu+\nu+2l+1)}{(2\chi)^{\mu+\nu+2l-1} l! \Gamma(\mu+l+1) \Gamma(\nu+l+1) \Gamma(\mu+\nu+l+1)}. \quad (7)$$

This series is found to converge for $x>2$. Hence, (7) gives a series solution of the integrals for large separation. The expression which is valid for x less than 2 can be derived as follows. The above equation is transformed into a contour integral given by

$$H(\mu, \nu; \chi) = \frac{1}{2\pi j} \cdot \int_{-j\infty}^{j\infty} \frac{\Gamma(-t) \Gamma(\mu+\nu+2t-1) \Gamma(\mu+\nu+2t+1)\, dt}{(2\chi)^{\mu+\nu+2t-1} \Gamma(\mu+t+1) \Gamma(\nu+t+1) \Gamma(\mu+\nu+t+1)}. \quad (8)$$

The contribution from the pole in the right-hand side of complex t-plane gives (7), while the contribution from the poles in the left-half plane gives the desired expression which is valid for $x<2$ [6]. It is readily found that simple poles are at $t = -(\mu+\nu-1)/2$ and $t = -(\mu+\nu)/2 - l$ $(l = 0, 1, 2, \cdots, |\mu-\nu|/2)$, while double poles are at $t = -(\mu+\nu+1)/2 - l$ $(l = 0, 1, 2, \cdots)$. The residues at these poles are given by

$$\begin{aligned} H(\mu, \nu; \chi) = {} & \frac{4}{\pi} \frac{\sin\left\{\frac{\mu-\nu-1}{2}\pi\right\}}{\{(\mu+1)^2-\nu^2\}\{(\mu-1)^2-\nu^2\}} - \frac{\chi}{2\mu}\delta_{\mu\nu} - \frac{1}{8}(-1)^{|\mu-\nu/2} \\ & \cdot \sum_{l=1}^{\frac{|\mu-\nu|}{2}} (-1)^l \frac{\Gamma\left(\frac{\mu+\nu}{2}+l\right) \Gamma\left(\frac{|\mu-\nu|}{2}+l\right) (2\chi)^{2l+1}}{\Gamma\left(\frac{|\mu-\nu|}{2}-l+1\right) \Gamma\left(\frac{\mu+\nu}{2}-l+1\right) \Gamma(2l+2) \Gamma(2l)} + \frac{1}{8} \\ & \cdot \sum_{l=0}^{\infty} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}+l\right)(2\chi)^{2l+2}}{\Gamma\left(\frac{\mu-\nu+1}{2}-l\right) \Gamma\left(\frac{\nu-\mu+1}{2}-l\right) \Gamma\left(\frac{\mu+\nu+1}{2}-l\right) \Gamma(2l+3) \Gamma(2l+1)} \left\{-\psi\left(\frac{\mu+\nu+1}{2}+l\right)\right. \\ & \left. -\psi\left(\frac{\mu-\nu+1}{2}-l\right) - \psi\left(\frac{\nu-\mu+1}{2}-l\right) - \psi\left(\frac{\mu+\nu+1}{2}-l\right) - 2\log 2\chi + 2\psi(2l+3) + 2\psi(2l+1)\right\} \end{aligned} \quad (9)$$

where $\psi(x) = d\Gamma(x)/dx$ is the di-gamma function.

Using the function defined by (6), the matrix elements given in (4) can be derived as follows. $K(n, m; \epsilon)$ can be expressed as

$$\begin{aligned} K(n, m; \epsilon) &= \int_0^{\infty} \frac{\sinh q\xi}{\sinh q\xi + \epsilon_r \cosh q\xi} \frac{J_m(\xi) J_n(\xi)}{\xi^2} d\xi \\ &= \frac{1}{\epsilon_r + 1} \sum_{l=0}^{\infty} (-P)^l \{H[n, m; 2lq] - H[n, m; (2l+2)q]\} \end{aligned} \quad (10)$$

where $P = (\epsilon_r - 1)/(\epsilon_r + 1)$. As a special case of parallel capacitor in an empty space, $K(n, m; \epsilon_0)$ simplifies to

$$K(n, m; \epsilon_0) = \tfrac{1}{2}[H(n, m; 0) - H(n, m; 2q)]. \quad (11)$$

For small q, the approximate expressions for $K(n, m; \epsilon_0)$ are readily derived using (9). The expansions up to order q^4 for small separation are given by

$$K(2n+1,2m+1;\epsilon_0)=\frac{q}{4n+2}\delta_{nm}-(-1)^{m-n}\cdot\frac{q^2}{\pi}\left\{\log 4-\log q-\frac{1}{2}-\sum_{l=1}^{|m-n|}\frac{1}{l-1/2}-\sum_{l=1}^{m+n}\frac{1}{l+1/2}\right\}$$
$$-(-1)^{m-n}\frac{2}{3}(m+n+1)(|m-n|)q^3+(-1)^{m-n}\cdot\frac{q^4}{3}\left(m+n+\frac{3}{2}\right)\left(m+n+\frac{1}{2}\right)\left(n-m+\frac{1}{2}\right)\left(n-m-\frac{1}{2}\right)$$
$$\cdot\left[-\frac{1}{2}\left\{\sum_{l=1}^{|n-m+1|}+\sum_{l=1}^{|n-m-1|}+\sum_{l=1}^{m+n+2}+\sum_{l=1}^{m+n}\right\}\frac{1}{l-1/2}+\log 4-\log q+\frac{43}{12}\right]+0(q^5). \quad (12)$$

For rather large values of q, we should return to the series expression given by (9) and (11), which are convenient for numerical calculation using an electronic computer. The validity of approximate expressions for $K(2n+1,\,2m+1;\,\epsilon_0)$ is verified numerically. In Fig. 1, we present numerical results of $K(2n+1,\,2m+1;\,\epsilon_0)$ for rather small order (n,m) when they are approximated by the first two terms (solid line) and by the first three terms (dotted line). The corresponding exact results are shown by broken lines. From these figures it is found that the approximate expressions truncated by the first two terms and by the first three terms for diagonal elements give precise results even for rather large values of q, while those for off-diagonal elements depart from the exact results for very small values of q, though the range of q in which the first three terms approximation is valid expand compared with that for the first two terms approximation. So, for off-diagonal elements we must rely on rigorous series expressions given in (9) and (11), or on numerical integration instead of the approximated expansion even for very small values of q.

The expression of $K(2n+1,\,2m+1;\,\epsilon)$ when the capacitor is filled with arbitrary dielectrics can be calculated using (10). The approximate expansion of $K(2n+1,\,2m+1;\,\epsilon)$ for small q could be derived similarly, though we are required to carry out one more summation. Chew and Kong have derived the first two terms of $K(2n+1,\,2m+1;\,\epsilon)$ for small q using a quite different method. But, as mentioned above, we prefer to calculate directly from (9) and (10) for any values of q practically.

A numerical calculation for capacitance based on the series expression for $K(m,n,\epsilon)$ given in (10) is carried out for $\epsilon_r=1$ and $\epsilon_r=2.65$. The results are shown in Table I with those by other methods. Results for column 1 and 2 represent the calculated results by the present method taking into account 1 and 10 unknowns A_n, respectively. Column 3 shows the results calculated from another expression drived from (1) and (2) with $\kappa=1/2$. Columns 4–6 show the results borrowed from the paper by Chew and Kong. It is noted from the tables that one-term approximation, in which A_0 is determined from $A_0 = E_0/K(1,1,\epsilon)$ in (3), gives surprisingly accurate results even for a large separation if $K(1,1,\epsilon)$ is calculated exactly. The error of one-term approximation is roughly 10% for large separation of $d/a=1$ independent of dielectric constant of substrate. The numerical results for a very small separation seem to be new.

III. Conclusion

We have derived a rigorous series expression for the integrals which appear in the study of a circular parallel capacitor when the problem is formulated by the method of dual integral equa-

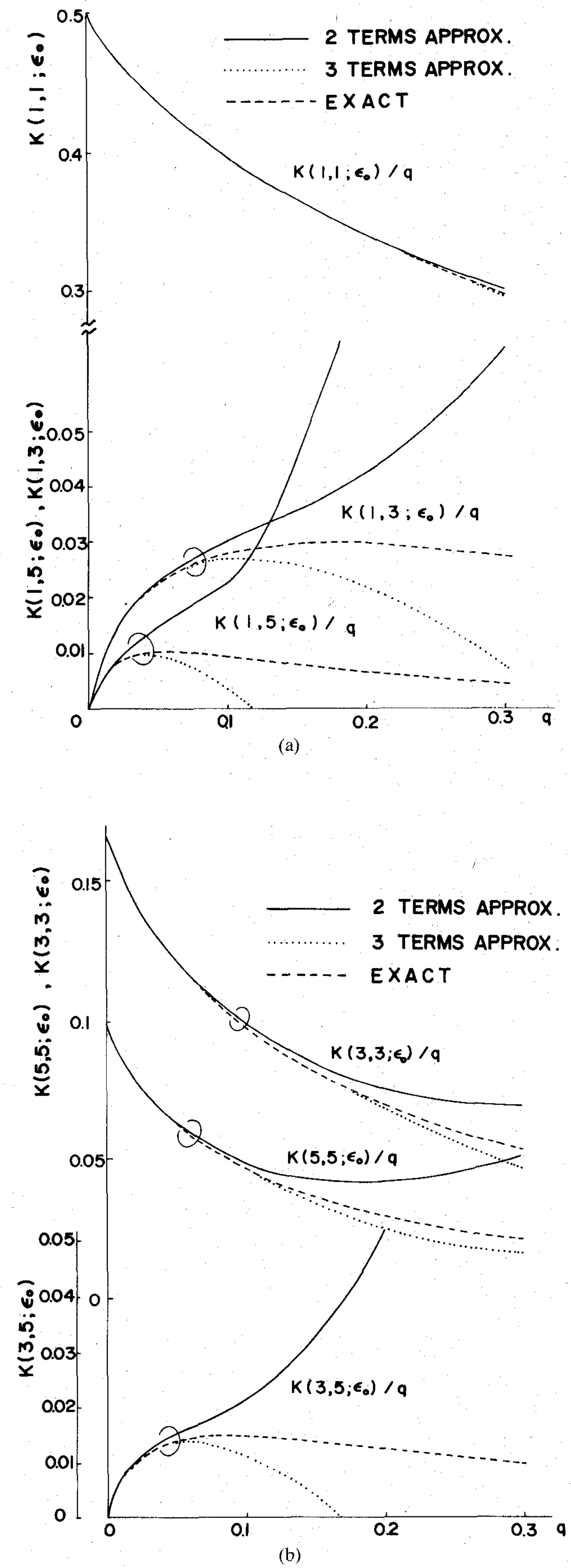

Fig. 1. Validity of approximate expressions for $K(n,m;\epsilon_0)$. The broken lines are corresponding exact solutions. (a) $K(1,1;\epsilon_0)$, $K(1,3;\epsilon_0)$, $K(1,5;\epsilon_0)$. (b) $K(3,3;\epsilon_0)$, $K(3,5;\epsilon_0)$, $K(5,5;\epsilon_0)$.

TABLE I
CAPACITANCE OF MICROSTRIP CIRCULAR DISK

ε_r=1.0

	1	2	3	4	5	6
d/a	one term approx	10 terms approx	another expression K=1/2	NM	SNA	ALB
0.01	1.0362	1.0421				
0.02	1.0651	1.0778				
0.03	1.0916	1.1110				
0.04	1.1168	1.1426				
0.05	1.1411	1.1730	1.1756			
0.06	1.1647	1.2026	1.2052			
0.07	1.1878	1.2315	1.2342			
0.08	1.2106	1.2599	1.2626			
0.09	1.2330	1.2877	1.2905			
0.1	1.2552	1.3115	1.3180	1.317	1.32	1.20
0.2	1.4702	1.5769	1.5800	1.580	1.57	1.32
0.3	1.6814	1.8265	1.8300	1.830	1.82	1.40
0.4	1.8935	2.0714	2.0751	2.0751	2.06	1.46
0.5	2.1075	2.3142	2.3183	2.3183	2.32	1.50
0.6	2.3239	2.5565	2.5608	2.5607	2.59	1.53
0.7	2.5425	2.7988	2.8034	2.8034	2.88	1.55
0.8	2.7631	3.0415	3.0464	3.0464	3.16	1.57
0.9	2.9855	3.2850	3.2901	3.2901	3.48	1.57
1.0	3.2095	3.5292	3.5346	3.5346	3.81	1.56

ε_r=2.65

	1	2	3	4	5	6
d/a	one term approx	10 terms approx	another expression K=1/2	NM	SNA	ALB
0.01	1.0179	1.0210				
0.02	1.0328	1.0404				
0.03	1.0468	1.0590				
0.04	1.0602	1.0769				
0.05	1.0734	1.0945	1.0969			
0.06	1.0862	1.1117	1.1142			
0.07	1.0989	1.1286	1.1311			
0.08	1.1114	1.1454	1.1479			
0.09	1.1239	1.1620	1.1645			
0.1	1.1362	1.1784	1.1809	1.180	1.18	1.14
0.2	1.2588	1.3380	1.3408	1.341	1.33	1.25
0.3	1.3836	1.4943	1.4973	1.497	1.48	1.35
0.4	1.5127	1.6502	1.6533	1.6533	1.64	1.43
0.5	1.6463	1.8066	1.8100	1.8100	1.80	1.51
0.6	1.7839	1.9642	1.9678	1.9678	1.97	1.59
0.7	1.9251	2.1232	2.1269	2.1269	2.15	1.66
0.8	2.0692	2.2835	2.2873	2.2873	2.33	1.73
0.9	2.2158	2.4451	2.4491	2.4491	2.52	1.80
1.0	2.3644	2.6081	2.6122	2.6122	2.72	1.86

tions or using Kobayashi potentials. The solutions for the integrals consist of two expressions in which one is valid for large separation and another for small separation. Using the series expressions derived here, the potential problem of circular parallel capacitor can be readily calculated. The integrals which occur in the study for line capacitance of parallel strip lines [7] can be performed using the present method and the result will be published in a forthcoming paper.

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The Measurement of the Electric Field Inside a Finite Dielectric Cylinder Illuminated by A Plane Wave

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***Abstract*—An experimental study of the distribution of the electric field induced inside a finite circular cylinder of water illuminated by an approximately plane electromagnetic wave is presented. The incident field was generated by using a monopole above the ground plane with a 90° corner reflector. The cylinder of water included a thin conducting tube at its center to shield the transmission lines leading to the probes. The graphs of selected measured distributions are displayed and interpreted. The measurements were carried out at 100, 300, and 600 MHz. The conductivity of the 50-cm long column of water was varied from approximately zero to 3.5 S/m. Both the amplitude and the phase of the induced electric field were measured in the experiment. Comparisons with a new theoretical solution developed by the authors are also included.**

I. INTRODUCTION

The interaction of electromagnetic radiation with a *finite* dielectric body is a problem of considerable practical interest. Such an investigation is of importance, for example, in the assessment of the biomedical hazards of EM radiation, the study of antennas attached to dielectric or poorly conducting aircraft, and the use of transponders embedded in biological organisms. The major focus of research in this area has been on the development of numerical solutions of EM-field problems involving three-dimensional bodies [1]–[5]. On the experimental side, thermal probes [6], [7] have been employed successfully to determine temperature

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